Enhancement of spatiotemporal regularity in an optimal window of random coupling

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We investigate the spatiotemporal dynamics of a lattice of coupled chaotic maps whose coupling connections are dynamically rewired to random sites with probability p; namely, at any instance of time, with probability p a regular link is switched to a random one. In a range of weak coupling, where spatiotemporal chaos exists for regular lattices (i.e., for $p=0$), we find that $p>0$ yields synchronized periodic orbits. Further, we observe that this regularity occurs over a window of p values, beyond which the basin of attraction of the synchronized cycle shrinks to zero. Thus we have evidence of an optimal range of randomness in coupling connections, where spatiotemporal regularity is efficiently obtained. This is in contrast to the commonly observed monotonic increase of synchronization with increasing p, as seen, for instance, in the strong-coupling regime of the very same system.

I. INTRODUCTION

The dynamics of spatially extended systems has been a focus of intense research activity in the past two decades. In recent years it has become evident that modeling large interactive systems by finite-dimensional lattices on the one hand and fully random networks on the other is inadequate, as various networks, ranging from collaborations of scientists to metabolic networks, do not to fit in either paradigm [1]. Some alternate scenarios have been suggested, such as the small-world network [2]. Here one starts with a regular structure on a lattice—for instance, nearest-neighbor interactions. Then each regular link from a site is *rewired randomly* with probability p. This model is proposed to mimic real-life situations in which nonlocal connections exist along with predominantly local connections.

There is much evidence that random nonlocal connections, even in a small fraction, significantly affect geometrical properties, like the characteristic path length [3]. However, *its implications for dynamical properties are still unclear and even conflicting*. While the dynamics of coupled oscillators and coupled maps on regular lattices has been extensively investigated [4], there have been far fewer studies on the spatiotemporal features of nonlinear elements on more general network topologies [5,6].

Most existing case studies of coupled networks of dynamical elements indicate that features, such as degree of synchronization, vary *monotonically* with p. That is, it is observed that most dynamical properties interpolate between the limits of regular and random connections without in any sense being "optimal" or more pronounced at some intermediate value of p.

In this paper, however, we will provide evidence of a system where there exists a window of randomness where one obtains special dynamical features which cannot be anticipated from a simple interpolation between the regular and random limits. In particular, we will show the pronounced enhancement of spatiotemporal order in the system in an intermediate window of rewiring probability p. Our observations are markedly distinct from the commonly observed monotonic dependence of synchronization properties on p, as seen, for instance, in the strong-coupling regime of the very same system [6].

II. MODEL

Specifically we consider a one-dimensional ring of coupled logistic maps. The sites are denoted by integers $i=1,\ldots,N$, where N is the linear size of the lattice. On each site is defined a continuous-state variable denoted by $x_n(i)$, which corresponds to the physical variable of interest. The evolution of this lattice, under standard interactions with the nearest neighbors on either side, in discrete time n, is given by

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + (\epsilon/2)\{x_n(i+1) + x_n(i-1)\}. \quad (1)$$

The strength of coupling is given by ϵ. The local on-site map is chosen to be the fully chaotic logistic map, $f(x)=4x(1-x)$, as this map has widespread relevance as a prototype of low-dimensional chaos.

Now we will consider the above system with its coupling connections rewired randomly in varying degrees and try to determine what dynamical properties are significantly affected by the way connections are made between elements. In our study, at every update we will connect a site with probability p to randomly chosen sites and with probability $1-p$ to nearest neighbors. That is, with probability p the dynamical equations of the system are

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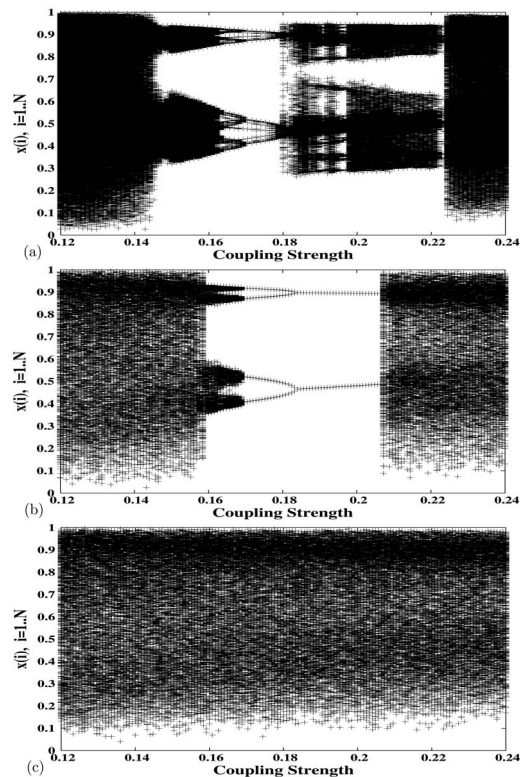[*]swarup_p@yahoo.com
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FIG. 1. Bifurcation diagram showing the state of the lattice $x_n(i)$, $i=1,\ldots,N$ ($N=50$), over $n=1,\ldots,25$, with respect to coupling strength ϵ, for (a) $p=0$, (b) $p=0.19$, and (c) $p=0.6$. The synchronized 2- and 4-cycles are clearly evident in (b), but not in (a) or (c).

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + (\epsilon/2)\{x_n(\eta) + x_n(\xi)\}, \qquad (2)$$

where ξ and η are random integers drawn from a uniform distribution in the interval $[1:N]$. With probability $1-p$ the dynamical equation of the nodes is given by Eq. (1). Notice that in the graph of the coupling connections, the number of edges of each vertex is unaltered by the rewiring (as opposed to certain models with "add-on" random links).

So at every instant a fraction p of randomly chosen nearest-neighbor links are replaced by random links. The case of $p=0$ corresponds to the usual nearest-neighbor interaction, while $p=1$ corresponds to completely random coupling.

We would like to underscore that the random connections are *dynamic* here, as the random links are switched around at every update. *This is in contrast to most studies which consider static (quenched) random connections.* Also notice that we switch the random links keeping all the geometric properties, such as clustering coefficient and mean path length, the same at all instants of time.

III. EMERGENCE OF SYNCHRONIZED CYCLES FROM SPATIOTEMPORAL CHAOS

We will now present numerical evidence that random rewiring has a pronounced effect on spatiotemporal order. The numerical results here have been obtained by sampling a large set of random initial conditions ($\sim 10^4$) and with lattice sizes ranging from 50 to 1000.
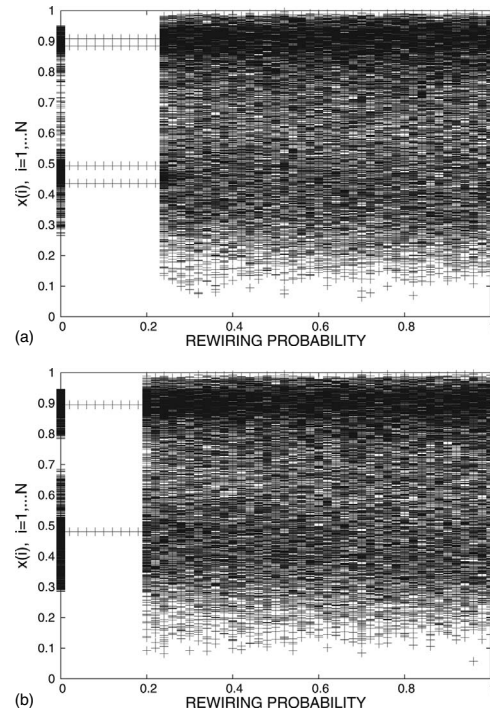

FIG. 2. Bifurcation diagram showing the state of the lattice $x_n(i)$, $i=1,\ldots,N$ ($N=50$), over $n=1,\ldots,25$, with respect to rewiring probability p, for coupling strengths (a) $\epsilon=0.18$ and (b) $\epsilon=0.2$. The synchronized 4- and 2-cycles are clearly evident in (a) and (b), respectively. Here transience time is 10^5.

Figure 1 displays the spatiotemporal state of the network, $x_n(i)$, $i=1,\ldots,N$, with respect to coupling strength ϵ, for the case of regular nearest-neighbor interactions (i.e., $p=0$) and for the case of random coupling with probability $p=0.19$ and $p=0.6$. It is clearly seen that the standard nearest-neighbor coupling does not yield regularity anywhere in the entire coupling range, while randomly rewiring with small probability $p=0.19$ creates a window in parameter space where *synchronized cycles gain stability.* So dynamic random coupling induces spatial order and, more remarkably, *temporal order* as well [7].

It is evident that *different periodicities* of the synchronized cycles are obtained under different coupling strengths, in the window of synchronization. Interestingly, however, when p is large this window of complete spatiotemporal regularity is lost again [see Figs. 1(c) and 2].

In order to quantify this phenomenon, we find the fraction of random initial conditions that get synchronized (after long transience). This provides a measure of the size of the basin of attraction of the synchronized state, denoted by B (see Fig. 3). When $B=1$ we obtain a global attractor for the synchronized cycles.

Figure 4 shows the synchronized basin size B with respect to rewiring probability p for different coupling strengths. It is clearly evident from the figure that for the regular lattice limit of $p=0$, the synchronized basin size is close to zero. As $p>0$ the basin for the synchronized state increases rapidly. For instance, for $p=0.01$ in Fig. 4, the synchronized state is the global attractor of the dynamics—i.e., $B=1$. However, as p exceeds an optimal window, the basin size decreases rap-
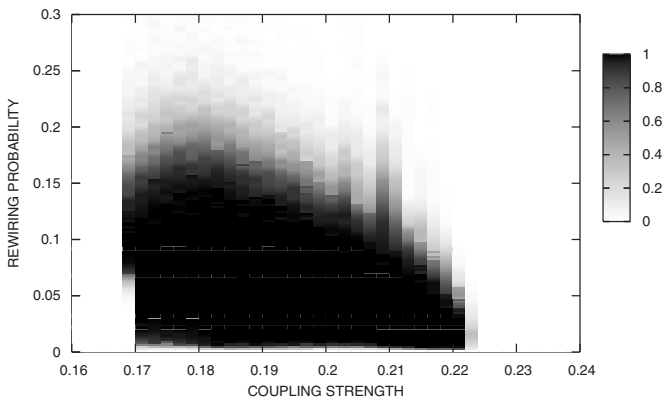
FIG. 3. Size of the basin of attraction of the synchronized cycles as a density plot in the space of coupling strength ϵ and random rewiring probability p for a system of size 100. Here transience is 10^4. Note that the grayscale in the figure shows the size of the basin of attraction B, ranging from white denoting $B=0$ to black denoting $B=1$.

idly, becoming zero again for high p. Thus *there exists a range of p values where synchronized cycles gain stability*. This observation is quantified in Fig. 5 which gives the range of rewiring probability, R, over which $B=1$ for different coupling strengths.

It is evident then that *there is a nonmonotonic enhancement of synchronization with increasing randomness in coupling connections* and synchronization is most enhanced in some window of p values. This is in sharp contrast to the monotonic increase in synchronization with increasing randomness in connectivity in the strong-coupling regime in the *same* system.

Note that the mean path length L drops rapidly to a small value as p increases from $p=0$ to $p\sim 1/N$. So the onset of regular behavior (at $p\rightarrow 0$ for large N) appears to be a consequence of small L. However, L quickly saturates to the limiting value, and so the existence of a window of regularity in p space cannot be determined by L alone.
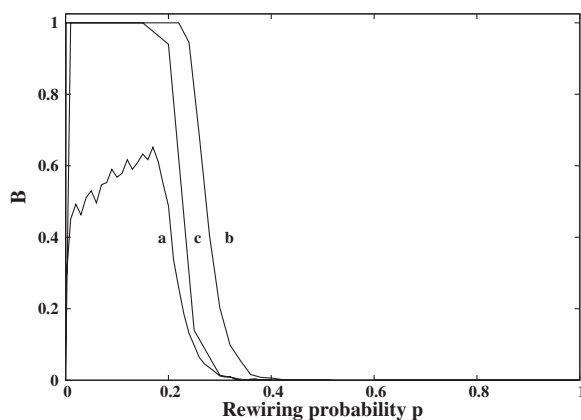

FIG. 4. Size of the basin of attraction of the synchronized state vs rewiring probability p for a system of size 100 for coupling strengths (a) $\epsilon=0.16$, (b) $\epsilon=0.19$, and (c) $\epsilon=0.20$. Here transience is 5×10^5.
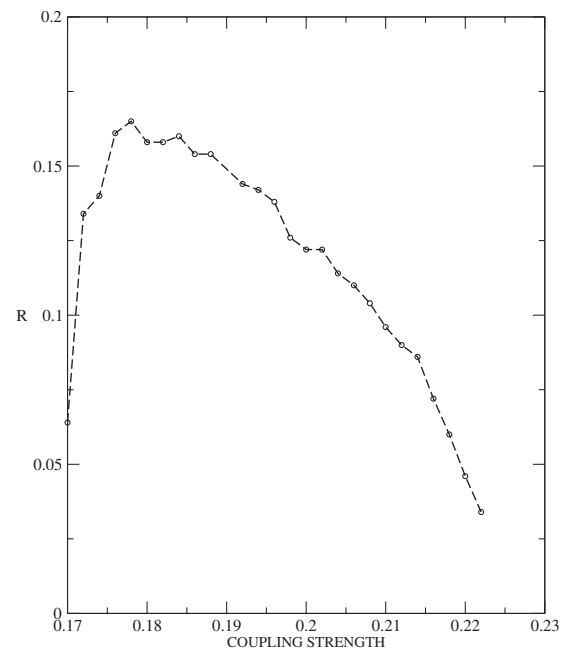

FIG. 5. Range of rewiring probability R, which yields a global attractor ($B=1$) versus coupling strength ϵ. Here system size is 100 and transience is 10^4.

IV. EFFICIENCY OF SYNCHRONIZATION

We calculate the average time taken for systems with random initial conditions to synchronize and denote this as $\langle T \rangle$. This quantity, which is a measure of the *efficiency of synchronization*, is displayed in Figs. 6 and 7.

It is evident from Fig. 6 that $\langle T \rangle$ first decreases slightly as p increases and then very sharply rises, by orders of magnitude, especially for larger lattices. It is clear then that synchronization is most efficient in some range of low p values. So in order to obtain fast and reliable synchronization it is a good strategy to choose low p, as this *ensures a global attractor for the synchronized cycles, as well as a fast approach to the synchronized state from a random initial state*.

As a consequence of the above, the size of the basin of attraction for the synchronized cycles, B, depends on the
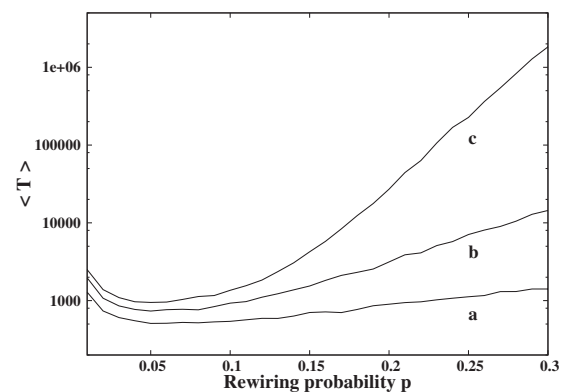

FIG. 6. Average time required to reach synchronization $\langle T \rangle$ vs rewiring probability p for coupling strength $\epsilon=0.19$ for system sizes $N=25,50,100$. The synchronized state is a global attractor for the system for the values of p displayed here.

allowed transience. Figure 7 shows the increase in the basin B as transience is increased. However, note that this increase is very slow, and increasing transience by two orders of magnitude changes the global attractor width in p space by a small amount.

Caveat: it appears possible, from the above observations, that in the limit of $T \to \infty$ a synchronized state may be reached for larger lattices as well. But it should be underscored that transience is always finite, and it is indeed very pertinent to know what generically happens in the limit of finite, but large $[\sim O(N^2)]$, transience, as is the case for the studies in this paper.

V. DISCUSSION

For regular coupling, standard linear stability analysis yields a window of stability in coupling parameter space for the 2- and 4-cycles, coinciding with that observed in Fig. 1(b). However, only initial states very close to the synchronized state are attracted to it, and so the basin of attraction of the synchronized cycle is vanishingly small for $p=0$ [8]. In fact, from generic random initial conditions here, one obtains a spatial state with domains of approximate synchronization, separated by regions of irregular "kinks."

Now such separated domains will not be stable under random links, as the random (nonlocal) connections will cross-couple such domains, leading to the formation of larger domains. A shorter mean path length L aids this trend, as different parts of the system get coupled. In fact, the only steady state possible under random coupling is one where all $x_n(i)$ are equal; namely, the system is spatially synchronized. So a small fraction of random links eliminates the kinks in the spatial profile.

However, there is also a countereffect. The stability of the local domains is aided by larger clustering coefficients C. So increasing p, giving rise to smaller clustering coefficients, has a destabilizing influence. So there is a competition between local effects dictated by C and global effects dictated by L, giving rise to this interesting phenomenon.
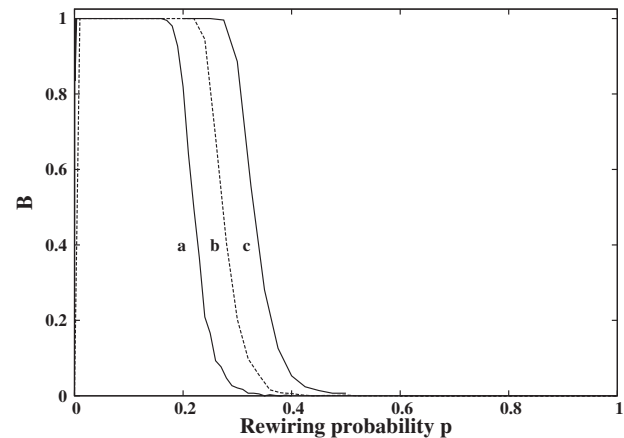

FIG. 7. Size of the basin of attraction of the synchronized state B vs rewiring probability p for coupling strength $\epsilon=0.19$ for a system of size 100 for maximum allowed transience equal to (a) 5 $\times 10^4$, (b) 5×10^5, and (c) 5×10^6.

In summary, we have investigated the spatiotemporal dynamics of a network of coupled chaotic logistic maps, with varying degrees of randomness in coupling connections. We find a window in coupling parameter space, in the weak-coupling regime, where random rewiring induces spatiotemporal order. Interestingly the basin of attraction for the synchronized cycles varies nonmonotonically with rewiring probability p. As p is increased, the basin of attraction of the synchronized state rapidly increases. At an intermediate (small) value of p the synchronized state becomes the global attractor of the system with (almost) all initial conditions being attracted to the synchronized state. However, interestingly, as p is further increased, the basin of attraction of the synchronized state shrinks to zero again. Thus we have strong evidence of the pronounced enhancement of spatiotemporal order in an intermediate window of p. This is in marked contrast to the monotonic increase in synchronization with increasing randomness in connectivity, observed in the strong-coupling regime of the same system.

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